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Research Article

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Analysis and Control of Anaerobic Digestion Dynamic Models

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Abstract

The anaerobic digestion process is highly nonlinear. This complexity arises from the interconnected, multi-stage biochemical reactions involving various microorganisms, influencing factors like pH, temperature, and substrate concentration. Understanding and controlling this nonlinearity is crucial for optimizing biogas production and wastewater treatment. Several factors must be considered, and multiple objectives must be met simultaneously. Bifurcation analysis and multi-objective nonlinear model predictive control (MNLMPC) calculations are performed on two dynamic models involving anaerobic digestion. The MATLAB program MATCONT was used to perform the bifurcation analysis. The MNLMPC calculations were performed using the optimization language PYOMO in conjunction with the state-of-the-art global optimization solvers IPOPT and BARON. The bifurcation analysis revealed the existence of branch points in both models. The branch points (which cause multiple steady-state solutions from a singular point) are very beneficial because they enable the Multiobjective nonlinear model predictive control calculations to converge to the Utopia point (the best possible solution) in the models. It is proved (with computational validation) that the branch points were caused because of the existence of two distinct separable functions in one of the equations in each dynamic model. A theorem was developed to demonstrate this fact for any dynamic model.

Keywords: Bifurcation, Optimization, Control, Anaerobic digestion

Background

Mailleret et al. investigated the robust regulation of anaerobic digestion processes. Mendez-Acosta et al. developed a robust feedforward/feedback control strategy for an anaerobic digester [1,2]. Shen et al. performed bifurcation and stability analysis of an anaerobic digestion model [3]. Rincon et al. researched the control of an anaerobic digester through the normal form of fold bifurcation [4]. Sbarciog et. al. determined appropriate operating strategies for anaerobic digestion systems [5]. Benyahia et al. conducted bifurcation and stability analysis of a two-step model for monitoring anaerobic digestion processes [6]. Rincón et al. performed a dynamic analysis for an anaerobic digester involving stability and bifurcation branches [7]. Lara-Cisneros et al. performed a dynamic optimization of methane production in anaerobic digestion via an extremum-seeking control approach [8]. Simeonov I et al. modelled anaerobic digestion with the production of hydrogen and methane [9]. Borisov et al. modelled the anaerobic digestion with hydrogen and methane production [10]. Chorukova et al. modelled the anaerobic digestion in a two-stage system with the production of hydrogen and methane, and three intermediate products [11]. This work aims to perform bifurcation analysis and multiobjective nonlinear control (MNLMPC) studies in two models involving the anaerobic digestion process, which are discussed in Rincón et al. (model 1), and Chorukova et al. (model 2) [7,11]. The paper is organized as follows. First, the model equations are presented, followed by a discussion of the numerical techniques involving bifurcation analysis and multiobjective nonlinear model predictive control (MNLMPC). The results are then presented, followed by the discussion and conclusions.

Anaerobic Digest Model Equations

The equations in the anaerobic digest model 1 are [7]

$$\begin{aligned}
\frac{dx_1}{dt} &= x_1(\mu_1 - (\alpha D)) \\
\frac{dx_2}{dt} &= x_2(\mu_2 - (\alpha D)) \\
\frac{ds_1}{dt} &= (s_{1,in} - s_1)D - k_1\mu_1 x_1 \\
\frac{ds_2}{dt} &= (s_{2,in} - s_2)D - k_3\mu_2 x_2 + k_2\mu_1 x_1 \\
\mu_1 &= \mu_{1,\max}\left(\frac{s_1}{ks1 + s_1}\right) \\
\mu_2 &= \mu_{2,\max}\left(\frac{s_2}{ks2 + s_2 + (s_2/ki)^2}\right)
\end{aligned} \qquad (1)$$

The parameter values are $\mu_{1,\max}$ =1.2;ks1=7.1; α = 0.5; $\mu_{2,\max}$ =0.74;ks2=9.28;ki=16, k1=10.53;k2=28.6;k3=1074, $s_{1,in}$=6.5;$s_{2,in}$=52. x_1 represents the concentration of acidogenic biomass, x_2 represents the concentration of methanogenic biomass, s_1 represents the concentration of organic substrate characterized by its chemical oxygen demand, and s_2 represents the concentration of volatile fatty acids (VFA). $\mu_{1,\max}$ $\mu_{2,\max}$ are the Monod coefficients α is the proportion of biomass not attached to the reactor ks1;ks2,ki, k1;k2;k3 are the saturation coefficients and , $s_{1,in}$;$s_{2,in}$ are the inlet concentrations. The equations in the anaerobic digest model 2Chorukova et al. [11] are

$$\begin{aligned}
\frac{ds_0}{dt} &= -(D1)(s_0) - \beta x1 s_0 + D1 yps_{0,in} \\
\frac{ds_1}{dt} &= -D1 * s_1 + \beta x1 s_0 - \frac{\mu_1 x_1}{y_1} \\
\frac{dx_1}{dt} &= x_1(\mu_1 - D1) \\
\frac{dp\mathrm{r}_1}{dt} &= \frac{x_1\mu_1}{Y_{pr,1}} - D1 p\mathrm{r}_1 \\
\frac{dbut_1}{dt} &= \frac{x_1\mu_1}{Y_{but,1}} - D1 but_1 \\
\frac{dac_1}{dt} &= \frac{x_1\mu_1}{Y_{ac,1}} - D1 ac_1 \\
\frac{d(xpr)}{dt} &= xpr(\mu_{pr} - D2) \\
\frac{dp\mathrm{r}_2}{dt} &= -\frac{xpr\,\mu_{pr}}{Y_{pr,2}} + D2(p\mathrm{r}_1 - p\mathrm{r}_2) \\
\frac{d(xbut)}{dt} &= xbut(\mu_{but} - D2) \\
\frac{d(but2)}{dt} &= -\frac{xbut\,\mu_{but}}{Y_{but,2}} + D2(but_1 - but_2)
\end{aligned}$$

$$\mu_1 = \mu_{1,\max}\left(\frac{s_1}{ks1 + s_1}\right)$$

$$\mu_{ac} = \mu_{ac,\max}\left(\frac{ac2}{ksac + ac2}\right)$$

$$\mu_{pr} = \mu_{pr,\max}\left(\frac{pr2}{kspr + pr2}\right)$$

$$\mu_{but} = \mu_{but,\max}\left(\frac{but2}{ksbut + but2}\right) \quad (2)$$

The parameter values are ks1=3.914; $\mu_{1,\max}$ =0.568; $\mu_{ac,\max} = 0.025$; ksac=0.8, $\mu_{pr,\max} = 0.05$; kspr=0.22; $\mu_{but,\max} = 0.05$; ksbut=0.22; β =1;Yp=1;Y1=0.08;Ypr1=4.2;Y_{but1}=2.1;Y_{ac1}=1.1;S_{0in}=40; Y_{pr2}=1.5;Y_{but2}=1.5;Y_{ac2}=0.5.

s_0 is the Cellulose concentration; s_1 the cellobiose substrate concentration, x_1 the acidogenic bacteria concentration, xpr the propionate degrading bacteria concentration; xbut is the Butyrate degrading bacteria concentration and xac the methanogenic bacteria concentration. pr1 and pr2 represent the Propionate concentration, but1 and but2 represent the butyrate concentration while ac1 and ac2 the acetate concentration. $\mu_{1,\max}$ $\mu_{ac,\max}$ $\mu_{pr,\max}$ represent the Monod coefficients while Ks1, ksac, kspr; ksbut are the saturation coefficients. Yp; Y1, Ypr1, Y_{but1}; Y_{ac1},Y_{pr2},Y_{but2},Y_{ac2} represent the Yield coefficients and S_{0in} is the inlet concentration. D1 and D2 are the dilution rates.

Bifurcation analysis

Bifurcation analysis involves multiple steady states and limit cycles. Multiple steady states occur because of the existence of branch and limit points. Hopf bifurcation points cause limit cycles. The MATLAB software MATCONT, which is used to perform the bifurcation calculations in this work, locates limit points, branch points, and Hopf bifurcation points (Dhooge Govearts, and Kuznetsov; Dhooge Govearts, Kuznetsov, Mestrom and Riet, Kuznetsov; and Govaerts [13-16]. This program detects Limit points (LP), branch points (BP), and Hopf bifurcation points(H) for an ODE system

$$\frac{dx}{dt} = f(x, \alpha) \quad (3)$$

$x \in R^n$ Let the bifurcation parameter be α. Since the gradient is orthogonal to the tangent vector,

The tangent plane at any point $z = [z_1, z_2, z_3, z_4, ...z_{n+1}]$ must satisfy

$$Az = 0 \quad (4)$$

Where A is

$$A = [\partial f / \partial x \quad | \partial f / \partial \alpha] \quad (5)$$

where $\partial f / \partial x$ is the Jacobian matrix. For both limit and branch points, the matrix $[\partial f / \partial x]$ must be singular. The n+1 th component of the tangent vector z_{n+1} = 0 for a limit point (LP) and at a branch point (BP), the matrix $\begin{bmatrix} A \\ z^T \end{bmatrix}$ must be singular. At a Hopf bifurcation point,

$$\det(2 f_x(x, \alpha) @ I_n) = 0 \quad (6)$$

@ indicates the bialternate product while I_n is the n-square identity matrix.

Multiobjective Nonlinear Model Predictive Control (MNLMPC)

For the MNLMPC calculations the procedure developed by Flores Tlacuahuaz et al. is used [17]. Here $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ (j=1, 2..n) represents the variables that need to be minimized/maximized simultaneously for a problem involving a set of ODE

$$\frac{dx}{dt} = F(x,u) \qquad (7)$$

n the total number of objective variables and t_f is the final time value. Let u be the control parameter vector. This MNLMPC procedure first solves the single objective optimal control problem independently optimizing each of the variables $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ individually. The minimization/maximization of $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ will lead to the values q_j^*. Then the optimization problem that will be solved is

$$\begin{aligned} &\min\left(\sum_{j=1}^{n}\left(\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) - q_j^*\right)\right)^2 \\ &subject\ to\ \frac{dx}{dt} = F(x,u); \end{aligned} \qquad (8)$$

This will provide the values of u at various times. The first obtained control value of u is implemented and the rest are discarded. This procedure is repeated until the implemented and the first obtained control values are the same or if the Utopia point where ($\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) = q_j^*$ for all j) is obtained. Pyomo (Hart et al.) is used for these calculations [18]. Here, the differential equations are converted to a Nonlinear Program (NLP) using the orthogonal collocation method The NLP is solved using IPOPT (Wächter And Biegler, and confirmed as a global solution with BARON (Tawarmalani, M. and N. V. Sahinidis) [19,20]. The algorithm can be summarized as follows

a) Optimize $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ and obtain q_j^* at various time intervals t_i. where subscript *i* is the index for each time step.

b) Minimize $\left(\sum_{j=1}^{n}\left(\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) - q_j^*\right)\right)^2$ and get the control values for various times.

c) Implement the first obtained control values
d) Repeat steps a to c until there is an insignificant difference between the implemented and the first obtained value of the control variables or if the Utopia point is achieved. The Utopia point is when $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) = q_j^*$ for all j.

Sridhar (2024)[21] proved that the MNLMPC calculations to converge to the Utopia solution when the bifurcation analysis reveals the presence of limit and branch points by applying the singularity condition to the co-state equation (Upreti, 2013)[22]. If the minimization of q_1 lead to the value q_1^* and the minimization of q_2 lead to the value q_2^* The MNLPMC calculations will minimize the function $(q_1 - q_1^*)^2 + (q_2 - q_2^*)^2$. The multiobjective optimal control (MOOC) will involve

$$\min \quad (q_1 - q_1^*)^2 + (q_2 - q_2^*)^2 \quad subject\ to\ \frac{dx}{dt} = F(x,u) \qquad (9)$$

If the objective function is differentiated we would obtain

$$\frac{d}{dx_i}((q_1-q_1^*)^2+(q_2-q_2^*)^2)=2(q_1-q_1^*)\frac{d}{dx_i}(q_1-q_1^*)+2(q_2-q_2^*)\frac{d}{dx_i}(q_2-q_2^*) \quad (10)$$

The Utopia point requires both $(q_1-q_1^*)$ and $(q_2-q_2^*)$ are zero. Therefore,

$$\frac{d}{dx_i}((q_1-q_1^*)^2+(q_2-q_2^*)^2)=0 \quad (11)$$

The co-state equation (Upreti; 2013)[22] is

$$\frac{d}{dt}(\lambda_i)=-\frac{d}{dx_i}((q_1-q_1^*)^2+(q_2-q_2^*)^2)-f_x\lambda_i; \quad \lambda_i(t_f)=0 \quad (12)$$

λ_i is the Lagrangian multiplier and t_f is the final time. The first term in this equation is 0 and hence

$$\frac{d}{dt}(\lambda_i)=-f_x\lambda_i;\lambda_i(t_f)=0 \quad (13)$$

At a limit or a branch point, for the set of ODE $\frac{dx}{dt}=f(x,u)$ f_x is singular. This implies that there are two different vector values for $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i)>0$ and $\frac{d}{dt}(\lambda_i)<0$. In between there should be a vector $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i)=0$. This, coupled with the boundary condition $\lambda_i(t_f)=0$ will lead to $[\lambda_i]=0$ This makes the problem an unconstrained optimization problem, and the only possible solution is the Utopia solution.

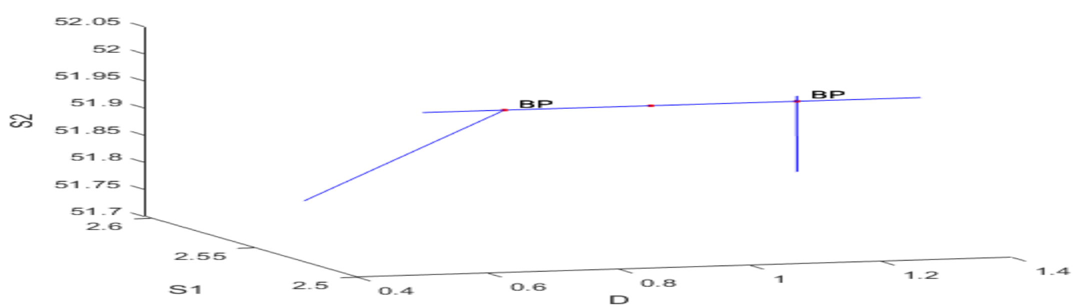

Figure 1a: Bifurcation analysis for anaerobic model 1 indicating the branch points

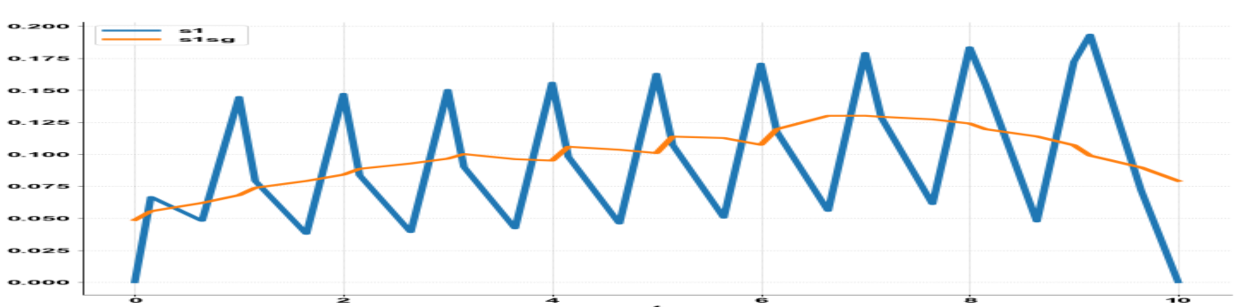

Figure 1b: MNLMPC model 1 , s1 with noise and s1sg (with Savitzky Golay Filter)   vs t

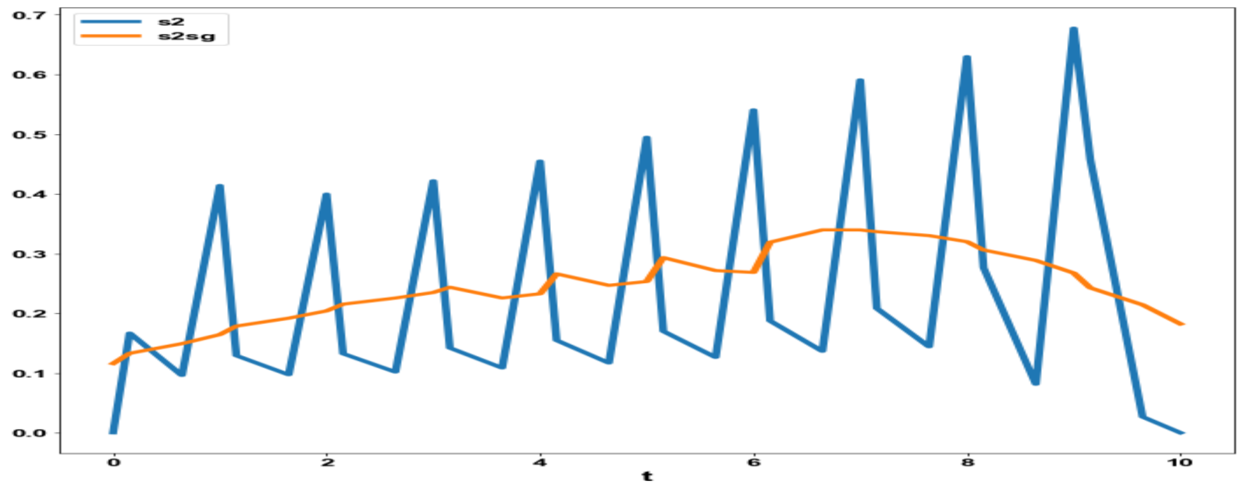

Figure 1c: MNLMPC model 1 , s2 with noise and s2sg (with Savitzky Golay Filter) vs t

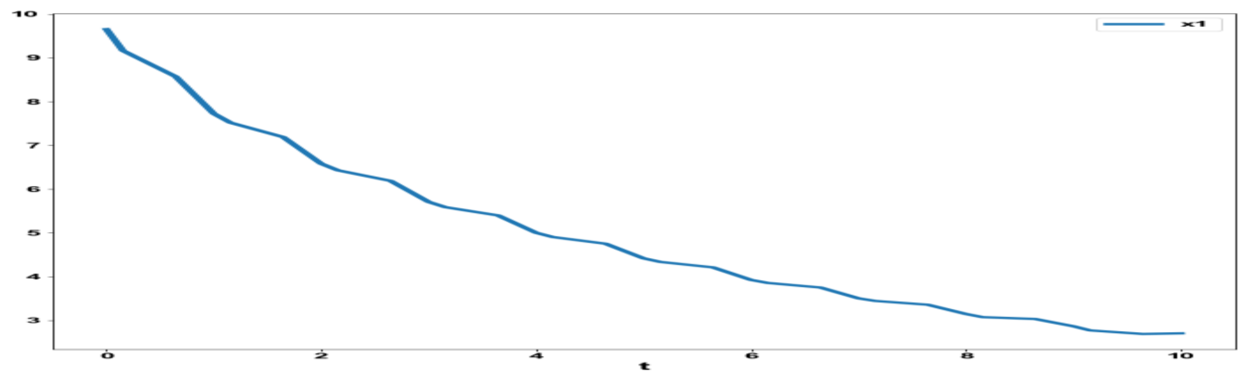

Figure 1d: MNLMPC model 1 , x1 vs t

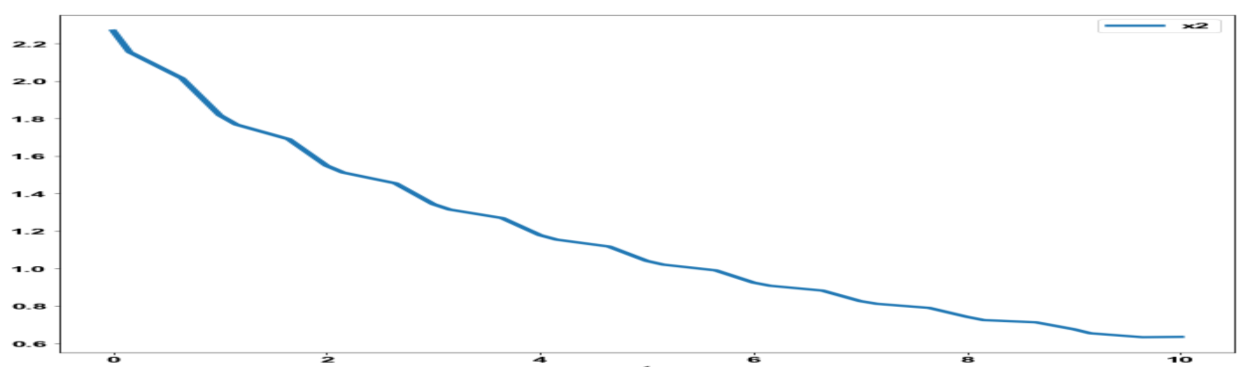

Figure 1e: MNLMPC model 1, x2 vs t

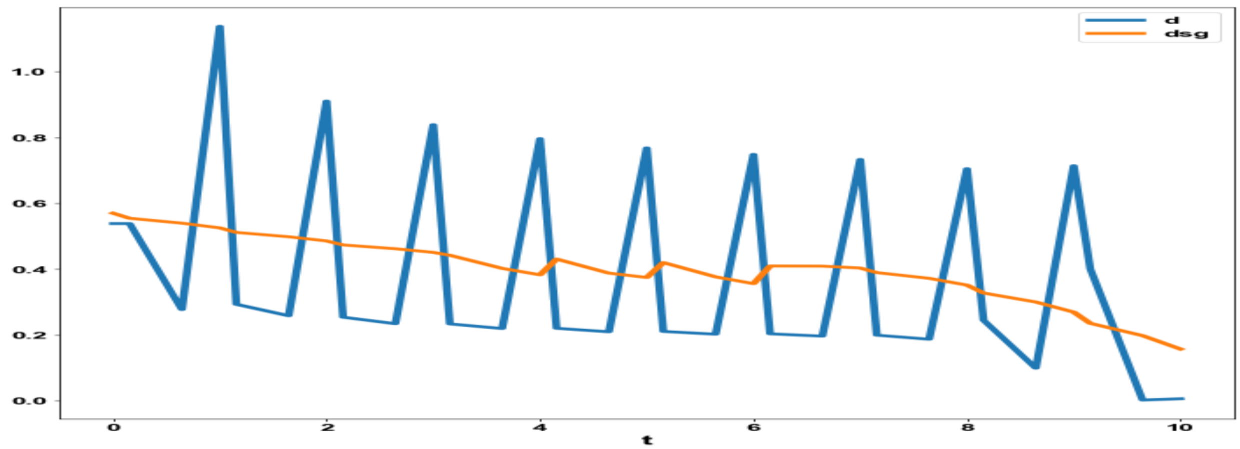

Figure 1f: MNLMPC model 1 , d with noise and dsg (with Savitzky Golay Filter) vs t

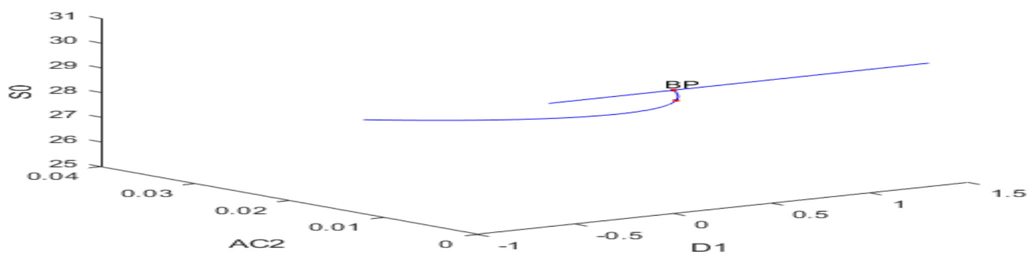

Figure 2a: Bifurcation analysis for anaerobic model 2 (D1 is the bifurcation parameter) indicating the branch point

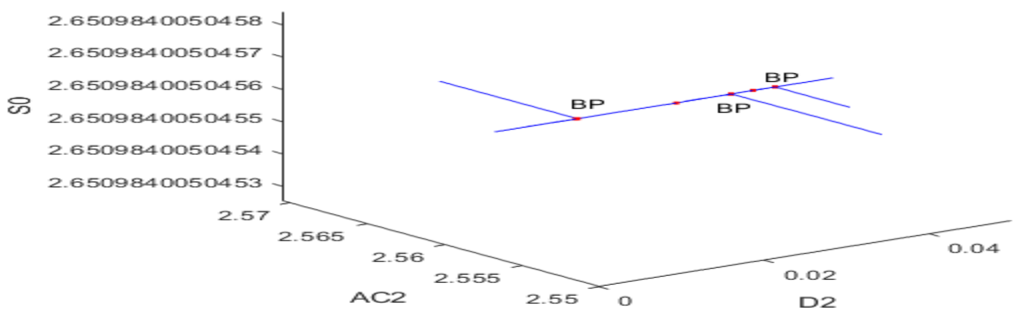

Figure 2b: Bifurcation analysis for anaerobic model 2 (D2 is the bifurcation parameter) indicating the branch point

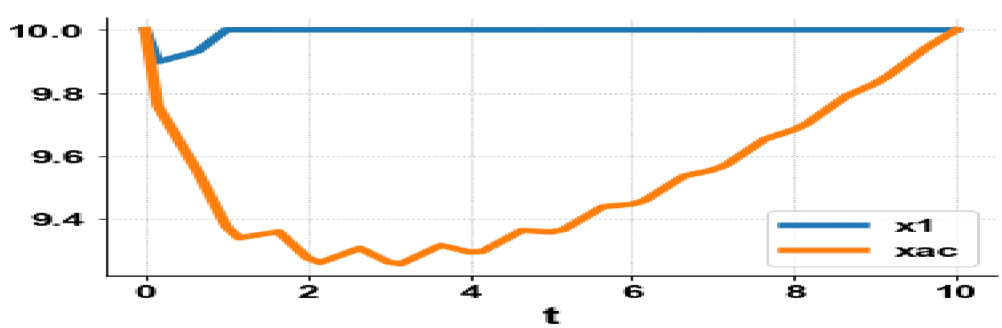

Figure 2c: MNLMPC model 2 , x1, xac vs t

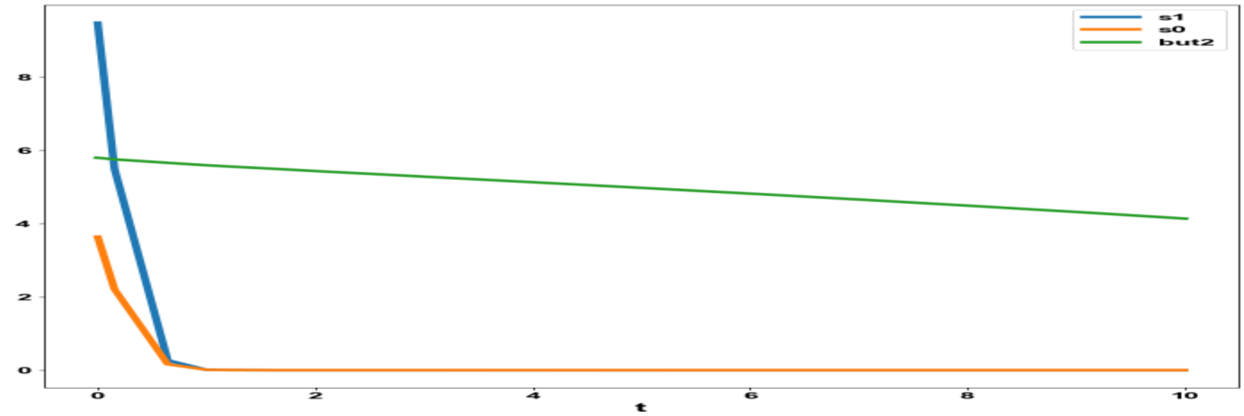

Figure 2d: MNLMPC model, s1, s0 but2   vs t

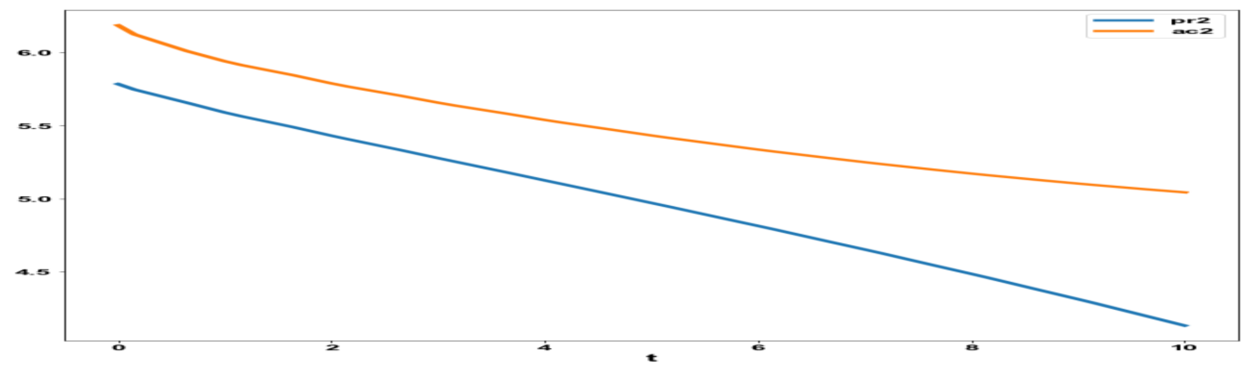

Figure 2e: MNLMPC model 2, pr2, ac2   vs t

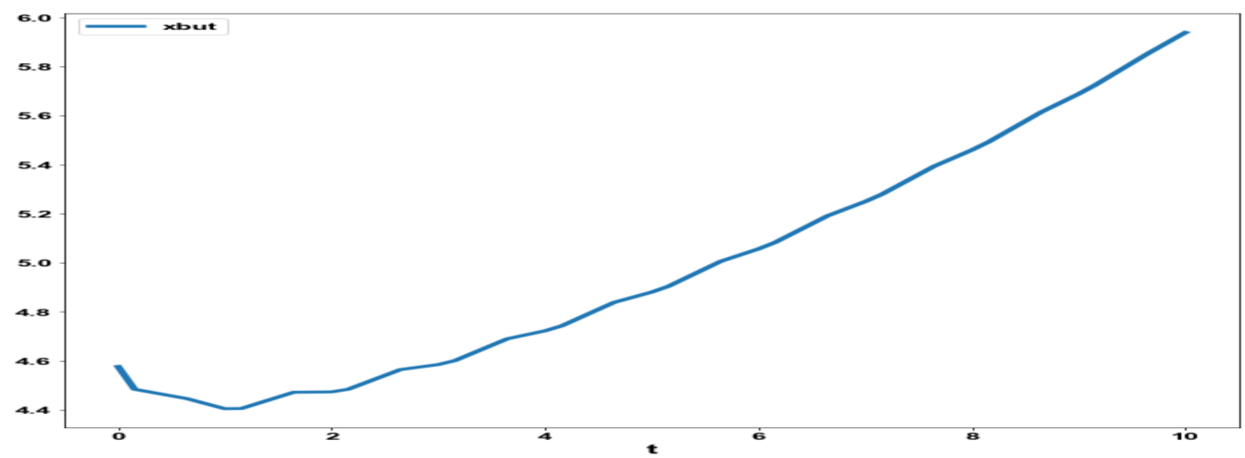

Figure 2f: MNLMPC model 2, xbut   vs t

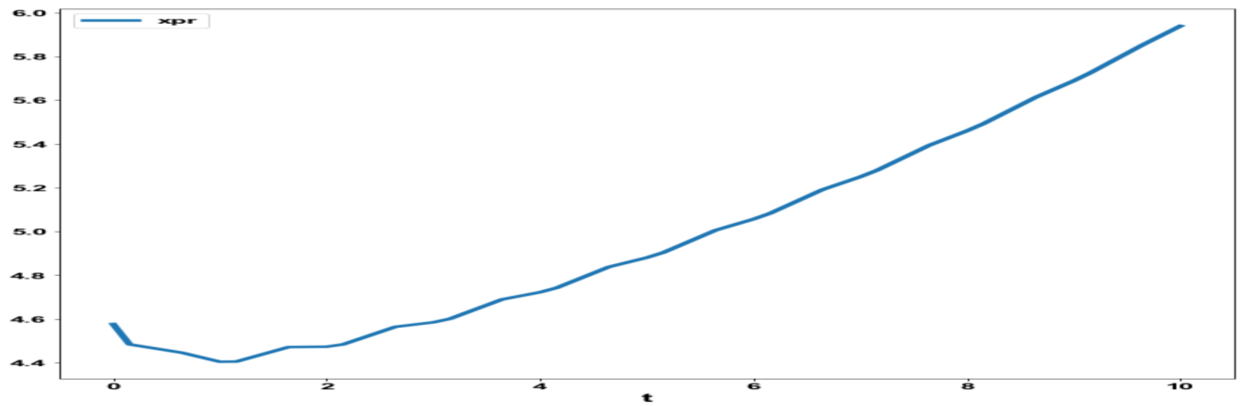

Figure 2g: MNLMPC model 2, xpr vs t

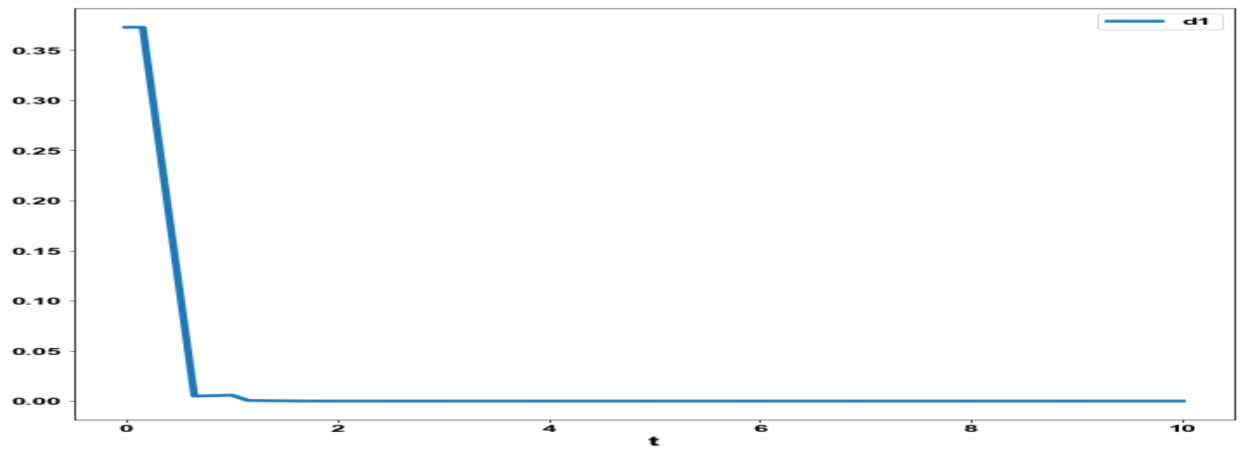

Figure 2h: MNLMPC model 2, d1 vs t

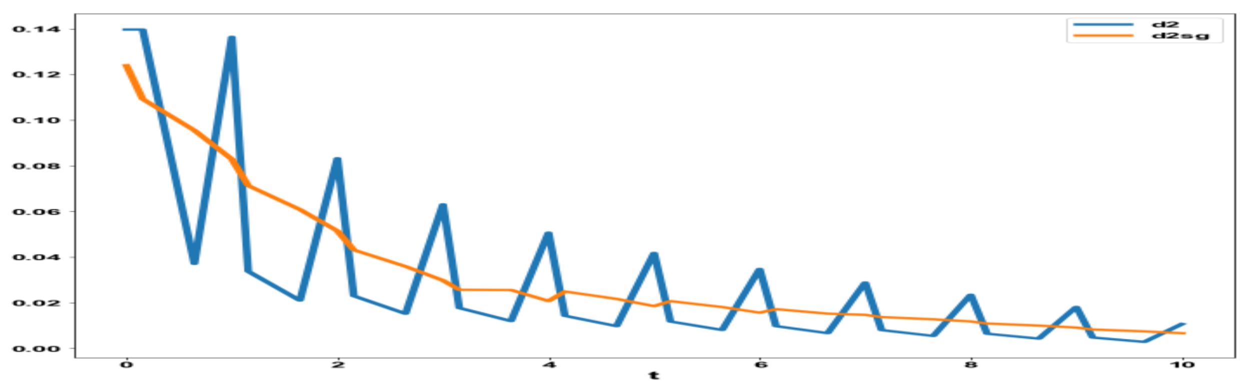

Figure 2i: MNLMPC model 2, d2 with noise and d2sg (with Savitzky Golay Filter) vs t

Results

In the anaerobic digest model 1, the bifurcation analysis with D as the bifurcation parameter, two branch points were located at (s_1, s_2, x_1, x_2, D) values of (0; 0.083947; 2.5; 6.920378; 0.625) and (0; 0; 2.5; 52; 1.071232). This is shown in Fig. 1a. For the MNLMPC calculations, $\sum_{t_{i=0}}^{t_i=t_f} s_1(t_i), \sum_{t_{i=0}}^{t_i=t_f} s_2(t_i)$ were minimized individually and led to values of 0 and 0. D was the control parameter. The multiobjective optimal control problem will involve the minimization of $(\sum_{t_{i=0}}^{t_i=t_f} s_1(t_i))^2 + (\sum_{t_{i=0}}^{t_i=t_f} s_2(t_i))^2$ subject to the equations governing Model 1. This led to a value of zero (the Utopia solution). The MNLMPC control value (D) was 0.5383. Figs 1b- 1f. show the various MNLMPC profiles. The profiles of s_1, s_2, and D exhibited a lot of noise, which were remedied using the Savitzky-Golay filter. Both the original and the modified profiles are shown together.

In the anaerobic digest model 2, the bifurcation analysis with D1 as the bifurcation parameter, a branch point was located at (s0;s1;x1;pr1;but1;ac1;xpr; pr2;xbut;but2;xac;ac2, D1) values of ((29.999857; 0.000143; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0.00002). This is shown in Fig. 2a. In the anaerobic digest model 2, the bifurcation analysis with D2 as the bifurcation parameter, three branch points were located at (s0;s1;x1;pr1;but1;ac1;xpr; pr2;xbut;but2;xac;ac2,D1) values of (2.650984; 2.127174; 2.817747; 0.670892; 1.341784; 2.561589; 0; 0.670892; 0; 1.341784 0;2.561589; 0.019050); (2.650984; 2.127174; 2.817747; 0.670892; 1.341784; 2.561589; 0;0.670892; 0; 1.341784; 0;2.561589 0.037653); (2.650984; 2.127174; 2.817747; 0.670892; 1.341784; 2.561589; 0; 0.670892; 0; 1.341784; 0; 2.561589 0.042957). This is shown in Fig. 2b. For the MNLMPC calculations, $\sum_{t_i=0}^{t_i=t_f} x_1(t_i), \sum_{t_i=0}^{t_i=t_f} xac(t_i)$ were maximized individually and led to values of 20 and 20. D1 and D2 were the control parameters. The multiobjective optimal control problem will involve the minimization of subject to the equations governing Model 2. This led to a value of zero (the Utopia solution). The MNLMPC control values of D1 and D2 were 3728 and 0.13962. Figures 2c- 2i. show the various MNLMPC profiles. The profiles of s1, s2, and D2 exhibited a lot of noise, which was remedied using the Savitzky-Golay filter. Both the original and the modified profiles are shown together.

Discussion

Theorem

A branch point singularity will occur in a dynamic system if and only if one of the functions in a dynamic system is separable into two distinct functions.

Proof

The proof consists of two parts. In the first part, the “if” condition will be proved. This will involve demonstrating that the presence of two distinct functions in an ODE set will lead to a branch point bifurcation. Consider a system of equations

$$\frac{dx}{dt} = f(x,\alpha) \tag{14}$$

α is the bifurcation parameter. $x \in R^n$. Define $y = [x,\alpha]$ and the matrix A as

$$A = [\frac{\partial f}{\partial y}] \tag{15}$$

The tangent at any point y; ($z = [z_1, z_2, z_3, z_4, ...z_{n+1}]$) must satisfy

$$Az = 0 \tag{16}$$

For a branch bifurcation point (Branch point from now on) (BP) the matrix $B = \begin{bmatrix} A \\ z^T \end{bmatrix}$ must be singular. Let any of the functions (in the set f(y)); f_i be separable into 2 functions ϕ_1, ϕ_2 as

$$f_i = \phi_1 \phi_2 \tag{17}$$

At steady-state $f_i(y) = 0$ and this will imply that either $\phi_1 = 0$ or $\phi_2 = 0$ or both ϕ_1 and ϕ_2 must be 0. This implies that two branches $\phi_1 = 0$ and $\phi_2 = 0$ will meet at a point where both ϕ_1 and ϕ_2 are 0.

At this point, the corresponding row in the matrix B would be

$$[\frac{\partial f_i}{\partial y_k}] \quad (18)$$

However,

$$[\frac{\partial f_i}{\partial y_k} = \phi_1(=0)\frac{\partial \phi_2}{\partial y_k} + \phi_2(=0)\frac{\partial \phi_1}{\partial y_k} = 0(\forall k = 1.,,n) \quad (19)$$

This implies that every element in the row $[\frac{\partial f_i}{\partial y_k}]$ would be 0, and hence the matrix B would be singular. The singularity in B implies that there exists a branch point.

In the second part, the " only if" condition will be proved and will involve demonstrating that the presence of a branch point requires that one of the functions must be separable into two distinct functions, each of which will be 0 at the branch point. For a branch point (BP) the matrix $B = \begin{bmatrix} A \\ z^T \end{bmatrix}$ must be singular. Any tangent at a point y that is defined by $z = [z_1, z_2, z_3, z_4, ...z_{n+1}]$) must satisfy

$$Az = 0 \quad (20)$$

For a branch point, there must exist two tangents at the singularity. Let the two tangents be z and w. This implies that

$$\begin{aligned} Az &= 0 \\ Aw &= 0 \end{aligned} \quad (21)$$

Consider a vector v that is orthogonal to one of the tangents (say z). v can be expressed as a linear combination of z and w ($v = \alpha z + \beta w$). Since $Az = Aw = 0$; $Av = 0$ and since z and v are orthogonal, $z^T v = 0$. Hence $Bv = \begin{bmatrix} A \\ z^T \end{bmatrix} v = 0$ which implies that B is singular. The vector v is orthogonal to z (a tangent). Any vector orthogonal to the tangent is the gradient of a function in the dynamic system $\frac{dx}{dt} = f(y)$. Hence, v **must be the gradient of a function** in the set of functions $f(x, \alpha)$. Let this function be f_j. The gradient of f_j is $\nabla f_j = \frac{\partial f_j}{\partial y_k} \forall\, k$. Hence

$$v = \nabla f_j = \frac{\partial f_j}{\partial y_k} \forall\, k \quad (22)$$

Multiplying B and v would result in the row $[\frac{\partial f_j}{\partial y_k}]$ multiplying the vector v which will result in $\sum (\frac{\partial f_j}{\partial y_k})^2; k = 1..n+1$ and the only way this can be zero is if each $(\frac{\partial f_j}{\partial y_k}) = 0$ implying that every element in the row is 0. At steady-state (say at a point y =b where b is a vector of dimension n+1); $f_j = 0$ and hence f_j can be represented as

$$f_j = (y - b) g_j \quad (23)$$

This implies that

$$f_j(y_k) = (y_k - b_k) g_j(y_k) \forall k \quad (24)$$

and

$$0 = (\frac{\partial f_j}{\partial y_k}) = g_j + (y_k - b_k)\frac{\partial g_j}{\partial y_k} \tag{25}$$

Since $y_k = b_k$ this would imply that $g_j = 0$

Hence $f_j = 0$ would imply that at a branch point, it can be split into two equations

$$\begin{aligned} &(y-b) = 0 \\ &g_j = 0 \end{aligned} \tag{26}$$

This completes the proof.

In the anaerobic digest model 1, 2 branch point was located at values of (x_1, x_2, s_1, s_2, D) values of (0; 0, 2.5; 52 ; 0.625) and (0; 0; 2.5; 52; 1.071232). At these points, all the variables are the same except the values of D. The two different values of D are 0.625 and 1.071232.

The first ODE in the anaerobic digest model 1 that will be considered is

$$\frac{dx_1}{dt} = x_1(\mu_1 - (\alpha D)) \tag{27}$$

Here, the two distinct equations are

$$\begin{aligned} &(\mu_1 - (\alpha D)) = 0 \\ &x_1 = 0 \end{aligned} \tag{28}$$

With x_1=0; s1=2.5; $\mu_{1,\max}$ =1.2;ks1=7.1; $\alpha = 0.5;$ D=0.625 and using the formula

$$\mu_1 = \mu_{1,\max}(\frac{s_1}{ks1 + s_1}) \tag{29}$$

We get both $(\mu_1 - (\alpha D)) = 0; x_1 = 0$ the distinct equations are satisfied, validating the theorem.

The second ODE in the anaerobic digest model 1 that will be considered is

$$\frac{dx_2}{dt} = x_2(\mu_2 - (\alpha D)) \tag{30}$$

Here, the two distinct equations are

$$\begin{aligned} &(\mu_2 - (\alpha D)) = 0 \\ &x_2 = 0 \end{aligned} \tag{31}$$

With x_2 =0; s2=52; $\mu_{2,\max}$ =0.74;ks2=9.28;ki=16 $\alpha = 0.5;$ D=1.071232 and using the formula

$$\mu_2 = \mu_{2,\max}(\frac{s_2}{ks2 + s_2 + (s_2 / ki)^2}) \tag{32}$$

We get both $(\mu_2 - (\alpha D)) = 0; x_2 = 0$ the distinct equations are satisfied, validating the theorem.

In the anaerobic digest model 2 with D1 as bifurcation parameter, a branch point was located at values of (s0;s1;x1;pr1;but1;ac1;xpr;pr2;xbut;but2;xac;ac2,D1) values of (29.999857; 0.000143; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0.00002). The two distinct equations can be obtained from the first ODE in the anaerobic digest model 2

$$\frac{dx_1}{dt} = x_1(\mu_1 - D1) \tag{33}$$

Here, the two distinct equations are

$$\begin{aligned} &(\mu_1 - D1) = 0 \\ &x_1 = 0 \end{aligned} \tag{34}$$

With x_1=0; s_1=0.000143;ks1=3.914; $\mu_{1,\max}$ =0.568; D1=2.0e-05; and using the formula

$$\mu_1 = \mu_{1,\max}\left(\frac{s_1}{ks1 + s_1}\right) \tag{35}$$

we get $\mu_1 = D1 = 2.e-05$ both the distinct equations are satisfied, validating the theorem.

In the anaerobic digest model 2 with D2 as bifurcation parameter, three branch points were found at (s0;s1;x1;pr1;but1;ac1;xpr;pr2;xbut;but2;xac;ac2,D1) values of (2.650984; 2.127174; 2.817747; 0.670892; 1.341784; 2.561589; 0; 0.670892; 0; 1.341784 0;2.561589; 0.019050); (2.650984; 2.127174; 2.817747; 0.670892; 1.341784; 2.561589; 0;0.670892; 0; 1.341784; 0;2.561589 0.037653); and (2.650984; 2.127174; 2.817747; 0.670892; 1.341784; 2.561589; 0; 0.670892; 0; 1.341784; 0; 2.561589 0.042957); At these points, all the variables are the same except the values of D2. The three different values of D2 are 0.019050, 0.037653, and 0.042957.

The first differential equation that will be considered is

$$\frac{d(xac)}{dt} = xac(\mu_{ac} - D2) \tag{36}$$

The two distinct equations that can be obtained from this ODE are

$$(\mu_{ac} - D2) = 0 \tag{37}$$

and

$$xac = 0 \tag{38}$$

Since

$$\mu_{ac} = \mu_{ac,\max}\left(\frac{ac2}{ksac + ac2}\right) \tag{39}$$

with ac2=2.561589 and $\mu_{ac,\max} = 0.025;$ and, ksac=0.8; we get the value of $\mu_{ac} =$ 0.019050, which is the same as the first D2 value satisfying both the distinct equations validating the theorem. The second differential equation that will be considered is

$$\frac{d(xpr)}{dt} = xpr(\mu_{pr} - D2) \tag{40}$$

The two distinct equations that can be obtained from this ODE are

$$(\mu_{pr} - D2) = 0 \tag{41}$$

and

$$xpr = 0 \tag{42}$$

Since

$$\mu_{pr} = \mu_{pr,\max}\left(\frac{pr2}{kspr + pr2}\right) \tag{43}$$

with pr2=0.670892 and $\mu_{pr,\max} = 0.05;$ and, kspr=0.22; we get the value of $\mu_{pr} =$ 0.037653, which is the same as the second D2 value satisfying both the distinct equations validating the theorem. The third differential equation that will be considered is

$$\frac{d(xbut)}{dt} = xbut(\mu_{but} - D2) \tag{44}$$

The two distinct equations that can be obtained from this ODE are

$$(\mu_{but} - D2) = 0 \tag{45}$$

and

$$xbut = 0 \tag{46}$$

Since

$$\mu_{but} = \mu_{but,\max}\left(\frac{but2}{ksbut + but2}\right) \tag{47}$$

with but2=1.341784 and $\mu_{but,\max} = 0.05;$ and, ksbut=0.22; we get the value of $\mu_{but} =$ 0.042957, which is the same as the third D2 value satisfying both the distinct equations validating the theorem. The MNLMPC calculations in both models converge to the Utopia solution, justifying the analysis of Sridhar [21].

Conclusion

Bifurcation analysis and multiobjective nonlinear control (MNLMPC) studies in two anaerobic digestion models. The bifurcation analysis revealed the existence and branch points in both models. The branch points (which cause multiple steady-state solutions from a singular point) are very beneficial because they enable the Multiobjective nonlinear model predictive control calculations to converge to the Utopia point (the best possible solution) in the models. It is proved (with computational validation) that the branch points were caused because of the existence of two distinct separable functions in one of the equations in each dynamic model. A theorem was developed to demonstrate this fact for any dynamic model. A combination of bifurcation analysis and Multiobjective Nonlinear Model Predictive Control(MNLMPC) for dynamic models involving anaerobic digestion is the main contribution of this paper.

Data Availability Statement

All data used is presented in the paper

Conflict of Interest

The author, Dr. Lakshmi N Sridhar, has no conflict of interest.

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References

1. Mailleret, L., Bernard, O., & Steyer, J. P. (2003). Robust regulation of anaerobic digestion processes. Water science and technology : a journal of the International Association on Water Pollution Research, 48(6), 87–94.
2. Méndez-Acosta, H. O., Campos-Delgado, D. U., Femat, R., & González-Alvarez, V. (2005). A robust feedforward/feedback control for an anaerobic digester. Computers & chemical engineering, 29(7), 1613-1623.
3. Shen, S., Premier, G. C., Guwy, A., & Dinsdale, R. (2007). Bifurcation and stability analysis of an anaerobic digestion model. Nonlinear Dynamics, 48(4), 391-408.
4. Rincon, A., Angulo, F., & Olivar, G. (2009). Control of an anaerobic digester through normal form of fold bifurcation. Journal of Process Control, 19(8), 1355-1367.
5. Sbarciog, M., Loccufier, M., & Noldus, E. (2010). Determination of appropriate operating strategies for anaerobic digestion systems. Biochemical engineering journal, 51(3), 180-188.
6. Benyahia, B., Sari, T., Cherki, B., & Harmand, J. (2012). Bifurcation and stability analysis of a two step model for monitoring anaerobic digestion processes. Journal of Process Control, 22(6), 1008-1019.
7. Rincón, A., Villa, J., Angulo, F., & Olivar, G. (2014). A dynamic analysis for an anaerobic digester: Stability and bifurcation branches. Mathematical Problems in Engineering, 2014(1), 514797.
8. Rincón, A., Villa, J., Angulo, F., & Olivar, G. (2014). A dynamic analysis for an anaerobic digester: Stability and bifurcation branches. Mathematical Problems in Engineering, 2014(1), 514797.
9. Simeonov, I., & Chorukova, E. (2016, June). Mathematical modeling of the anaerobic digestion with production of hydrogen and methane. In Proceedings of the 4th International Conference on Water, Energy and Environment (ICWEE), Burgas, Bulgaria (pp. 32-38).
10. Borisov, M., Dimitrova, N., & Simeonov, I. (2016). Mathematical modelling of anaerobic digestion with hydrogen and methane production. IFAC-PapersOnLine, 49(26), 231-238.
11. Chorukova, E., & Simeonov, I. (2020). Mathematical modeling of the anaerobic digestion in two-stage system with production of hydrogen and methane including three intermediate products. International Journal of Hydrogen Energy, 45(20), 11550-11558.
12. Dhooge, A., Govaerts, W., & Kuznetsov, Y. A. (2003). MATCONT: a MATLAB package for numerical bifurcation analysis of ODEs. ACM Transactions on Mathematical Software (TOMS), 29(2), 141-164.
13. Dhooge, A., Govaerts, W., Kuznetsov, Y. A., Mestrom, W., & Riet, A. M. (2003, March). Cl_matcont: a continuation toolbox in Matlab. In Proceedings of the 2003 ACM symposium on Applied computing (pp. 161-166).
14. Kuznetsov, Y. A. (1998). Elements of applied bifurcation theory. New York, NY: Springer New York.
15. Kuznetsov, Y. A. (2009). Five lectures on numerical bifurcation analysis. Utrecht University, Nl.
16. Govaerts, W. J. (2000). Numerical methods for bifurcations of dynamical equilibria. Society for Industrial and Applied Mathematics.
17. Flores-Tlacuahuac, A., Morales, P., & Rivera-Toledo, M. (2012). Multiobjective nonlinear model predictive control of a class of chemical reactors. Industrial & Engineering Chemistry Research, 51(17), 5891-5899.
18. Hart, W. E., Laird, C. D., Watson, J. P., Woodruff, D. L., Hackebeil, G. A., Nicholson, B. L., & Siirola, J. D. (2017). Pyomo-optimization modeling in python (Vol. 67, p. 277). Berlin: Springer.
19. Wächter, A., & Biegler, L. T. (2006). On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. Mathematical programming, 106(1), 25-57.
20. Tawarmalani, M., & Sahinidis, N. V. (2005). A polyhedral branch-and cut approach to global optimization. Mathematical programming, 103(2), 225-249.
21. Sridhar, L. N. (2024). Coupling Bifurcation Analysis and Multiobjective Nonlinear Model Predictive Control. Austin Chem Eng, 10(3), 1170.
22. Upreti, Simant Ranjan(2013); Optimal control for chemical engineers. Taylor and Francis.